

Generalized Finite Element Systems for smooth differential forms and Stokes' problem

Snorre H. Christiansen*, Kaibo Hu†

Abstract

We provide both a general framework for discretizing de Rham sequences of differential forms of high regularity, and some examples of finite element spaces that fit in the framework. The general framework is an extension of the previously introduced notion of Finite Element Systems, and the examples include conforming mixed finite elements for Stokes' equation. In dimension 2 we detail four low order finite element complexes and one infinite family of highorder finite element complexes. In dimension 3 we define one low order complex, which may be branched into Whitney forms at a chosen index. Much of the theory is developed in arbitrary dimension. The finite element spaces all consist of composite polynomials. The framework guarantees some nice properties of the spaces, in particular the existence of commuting interpolators. It also shows that some of the examples are minimal spaces.

MSC: 65N30, 58A12.

Introduction

This article is concerned with developing finite element spaces of differential forms with enhanced continuity properties, compared with those described in [16][3][1][8]. These spaces should be compatible in a precise sense, which in general will depend on the partial differential equation one wants to solve and will reflect the functional framework one adopts for the analysis. One way of phrasing compatibility is that the discrete spaces of differential forms should form a complex, equipped with bounded projections that commute with the exterior derivative.

In many cases such bounded projections can be obtained by an averaging technique [9][7], from the interpolators associated with degrees of freedom, defined on smooth differential forms. Since the averaging technique is defined so as to commute with the differential, we may concentrate on getting degrees of freedom that provide commuting interpolators. As it turns out, the existence of such degrees of freedom, on a finite element space, can be deduced from a few

*Department of Mathematics, University of Oslo, PO Box 1053 Blindern, NO 0316 Oslo, Norway. email: snorre@math.uio.no.

†Beijing International Center for Mathematical Research, Peking University, No. 5 Yiheyuan Road, Haidian District, Beijing 100871 China P.R. email: kaiboh@math.uio.no. On visit at University of Oslo.

algebraic constraints, that have been clarified in a framework of Finite Element Systems (FES) introduced in [4] and further developed in [5][8][6]. A precise notion of *compatibility* guarantees that the so-called harmonic degrees of freedom are unisolvent and provide an interpolator that commutes.

Recall Ciarlet's definition of a finite element (e.g. [10] §10), in terms of spaces equipped with degrees of freedom (DoF). The framework of FES gives DoFs a secondary role. Rather, compatibility is defined in terms of restrictions and differentials. On a compatible FES there will in general be *many* choices of DoFs, for the *same* spaces. The harmonic DoFs are a natural choice among these possibilities. As we will see DoFs seem most useful to describe low order elements, where there is not so much choice.

In this paper we provide both a generalization of the framework of FES that can handle higher order continuity of differential forms and some examples of spaces that fit into the framework. The generalization essentially consists in allowing for other types of restriction operators than pullback of differential forms. These restriction operators reflect that higher continuity implies that more information about the fields should be available on interfaces in the mesh. The examples of FES we provide, are all composite finite elements on a simplicial mesh, that are piecewise polynomials with respect to a simplicial refinement. For the spaces of scalar functions, considered as 0-forms, we use continuously differentiable composite elements, as introduced by Hsieh and Clough-Tocher and discussed further in [13][10][17][22]. The rest of the sequences, pertaining to differential k -forms for $k \geq 1$, appear to be new. These sequences end with conforming mixed finite elements for the Stokes equation: continuous vector-fields with either continuous or discontinuous divergence.

Our results are perhaps most closely related to those of [14], where finite element families for Stokes' equation are defined in 2D, for both $H^1 - L^2$ conforming and $H_{\text{div}}^1 - H^1$ conforming settings, as part of de Rham sequences with high regularity. In [19] these results are extended to 3D. The spaces attached to triangles or tetrahedra consist of polynomials, including polynomial bubbles. In particular they are smooth functions. Their degrees of freedom for vectorfields include in particular all first order partial derivatives at vertices. Our spaces of vectorfields, on the other hand, consist of composite polynomials which are not necessarily of class C^1 , and our degrees of freedom at vertices are just the vertex values and (for H_{div}^1) vertex values of the divergence, which is a particular combination of first order derivatives. We point out that our spaces come equipped with commuting interpolators. The lower order continuity/differentiability imposed at vertices (for instance), and the composite nature of our elements, seem important in this respect, from the point of view provided by FES, for the given continuity one wants to achieve. The commuting diagram that we obtain, as a consequence of compatibility, makes the proof of the inf-sup condition easier than the macro-element techniques introduced for Stokes in [21]. Or, at least, it provides an alternative type of proof.

To be more specific, we consider an n -dimensional domain S , say in the Euclidean space \mathbb{R}^n . For $r \geq 0$ we denote by $H^r \Lambda^k(S)$ the spaces of k -forms on S with partial derivatives up to order r in $L^2(S) \otimes \text{Alt}^k(\mathbb{R}^n)$. We denote by $H_d^r \Lambda^k(S)$ the following space:

$$H_d^r \Lambda^k(S) = \{u \in H^r \Lambda^k(S) : du \in H^r \Lambda^{k+1}(S)\}. \quad (1)$$

We are interested in the complexes:

$$\dots \longrightarrow H_d^r \Lambda^{k-1}(S) \longrightarrow H_d^r \Lambda^k(S) \longrightarrow H_d^r \Lambda^{k+1}(S) \longrightarrow \dots \quad (2)$$

We are also interested in letting r decrease in the complex, at some index, as follows:

$$\dots \longrightarrow H_d^r \Lambda^{k-1}(S) \longrightarrow H^r \Lambda^k(S) \longrightarrow H_d^{r-1} \Lambda^{k+1}(S) \longrightarrow \dots \quad (3)$$

If we restrict attention to dimension $n = 2$ and $r = 0, 1$ this leaves us with three possibilities:

$$H^1 \Lambda^0(S) \longrightarrow H_d^0 \Lambda^1(S) \longrightarrow H^0 \Lambda^2(S) \quad (4)$$

$$H^2 \Lambda^0(S) \longrightarrow H^1 \Lambda^1(S) \longrightarrow H^0 \Lambda^2(S) \quad (5)$$

$$H^2 \Lambda^0(S) \longrightarrow H_d^1 \Lambda^1(S) \longrightarrow H^1 \Lambda^2(S) \quad (6)$$

We refer to these sequences as de Rham sequences with regularity $(1, 0+, 0)$, $(2, 1, 0)$ and $(2, 1+, 1)$ respectively. The two last spaces in the two last sequences are of interest for conforming discretizations of the Stokes equation.

We are interested in constructing finite element spaces which provide sub-complexes of these complexes. These subcomplexes should be equipped with commuting interpolation operators. For this purpose a framework of FES has been developed for the first type of complex, starting in [4]. It is summarized in [8]. In this paper, we extend the framework so that it can encompass the other two types of complexes, and more generally, we believe, arbitrary $r \geq 0$ as well as switches between different r as sketched above. For small r we provide examples that illustrate that high order polynomials can be included in the finite element spaces, to achieve arbitrarily high approximation order. In arbitrary dimension we also illustrate that it may be useful to consider different simplicial refinements at different indices of the differential complex. Many more examples than those provided here, should fit in the proposed framework.

The paper can be seen as a step towards a general theory of discretization of highly continuous fields (sections of vector bundles), in terms of inverse systems of complexes of jets. From this point of view, the present paper provides examples of r -jets of order $r = 0$ and $r = 1$. This already seems adequate for many of the PDEs we have in mind, since they are at most second order.

The paper is organized as follows. In §1 we provide some tools for handling differential forms. In particular we characterize their inter-element continuity in terms of various restriction operators, and recall methods for proving sequence exactness under the exterior derivative. In §2 we provide four examples of low order composite finite element sequences in space dimension 2, both to motivate the framework of generalized finite element systems and to get the machinery started, with respect to higher order polynomials. In §3 we provide the appropriate notions on generalized FES, leading up to the notion of harmonic interpolator. In §4 we provide, in dimension 2, examples of composite finite element de Rham sequences with enhanced continuity and arbitrarily high degree of polynomials. In §5 we provide some tools for defining composite finite elements in arbitrary space dimension. In particular we define different simplicial

refinements and study some continuous piecewise affine forms on them. In §6 we provide a composite finite element de Rham sequence with enhanced continuity and low order polynomials (at most degree two). We also show how such sequences can be branched into Whitney forms at some index. We conclude with some topics for further research.

1 Preliminary tools for differential forms

Restriction operators adapted to different regularities. Consider a simplicial complex \mathcal{T} on a domain S in a vector space \mathbb{V} of dimension n . For differential forms which are piecewise smooth with respect to \mathcal{T} we have:

- $u \in H_d^0 \Lambda^k(S)$ iff the pullbacks to faces are singlevalued. If $T \in \mathcal{T}$ is a simplex, *pullback* means here pullback in the sense of differential forms by the injection $T \rightarrow S$. It remembers the action of u only on vectors which are tangent to T .
- $u \in H^1 \Lambda^k(S)$ iff the traces on faces are singlevalued. Here *trace* means restriction in the usual sense, remembering the action of u on all tangent vectors in S (not only T).
- $u \in H_d^1 \Lambda^k(S)$ iff the traces on faces of both u and du are singlevalued on faces. Here the word trace is used with the same meaning as above.

We interpret this by saying that we have defined three types of *restriction* operators. Explicitly, according to context, the restriction of a differential form $u \in C_d^r \Lambda^k(S)$ to a face T will be:

- the pullback of u , denoted $\text{pu}_T u$, which is in $C_d^r \Lambda^k(T)$.
- the trace of u , denoted $\text{tr}_T u$, which is in $C^r(T) \otimes \text{Alt}^k(\mathbb{V})$.
- the double-trace of u , written $(\text{tr}_T u, \text{tr}_T du)$, which is in $C^r(T) \otimes \text{Alt}^k(\mathbb{V}) \oplus C^r(T) \otimes \text{Alt}^{k+1}(\mathbb{V})$.

The framework of FES, introduced in [4] and developed further in [5][7][8][6] was designed to handle restrictions of the first type, whereas now we are interested in the other cases as well. More generally, we will consider a cellular complex \mathcal{T} and restrictions from T to T' where T, T' are cells in \mathcal{T} and $T' \subseteq T$.

Admissibility condition. When we start with a k -form $u \in C_d^r \Lambda^k(S)$, the trace of (u, du) on a cell T is in $C^r(T) \otimes \text{Alt}^k(\mathbb{V}) \oplus C^r(T) \otimes \text{Alt}^{k+1}(\mathbb{V})$, but all elements of the latter sum cannot occur. In other words there are admissibility conditions. In this paragraph we determine them.

Lemma 1.1. *Let \mathbb{V} be a finite dimensional vector space. Let (e_i) be a basis of \mathbb{V} and let (f_i) be the dual basis. Then for $u \in \text{Alt}^k(\mathbb{V})$, $k \geq 1$, we have:*

$$\sum_i f_i \wedge (u \lrcorner e_i) = k u. \quad (7)$$

Proof. By induction on k . □

We may consider that this identity is true also for $k = 0$, the left hand side being 0 by definition of contraction of 0-forms.

Proposition 1.2. *Fix $r \geq 0$. Let \mathbb{V} be a vector space and let T be a subspace. Let $v_0 \in C^{r+1}(T) \otimes \text{Alt}^k(\mathbb{V})$ and $v_1 \in C^r(T) \otimes \text{Alt}^{k+1}(\mathbb{V})$. The following are equivalent:*

- *There exists $u \in C^{r+1}\Lambda^k(\mathbb{V})$ such that $\text{tr}_T u = v_0$ and $\text{tr}_T du = v_1$.*
- *The induced forms $\text{pu}_T v_0 \in C^{r+1}\Lambda^k(T)$ and $\text{pu}_T v_1 \in C^r\Lambda^{k+1}(T)$ (obtained by remembering only the action on tangent vectors to T), are related by:*

$$d \text{pu}_T v_0 = \text{pu}_T v_1. \quad (8)$$

Proof. (i) The first condition implies the second, because the exterior derivative commutes with pullback.

(ii) We prove that the second condition implies the first. We write $\mathbb{V} = T \oplus U$. We introduce a vector field X on \mathbb{V} , defined by, for any $x \in T$ and any $y \in U$:

$$X(x + y) = y. \quad (9)$$

We choose a basis $(e_i)_{i \in I}$ of T and $(e_j)_{j \in J}$ of U . We impose $I \cap J = \emptyset$, so they combine to a basis of \mathbb{V} and we let $(f_i)_{i \in I \cup J}$ denote the corresponding dual basis of \mathbb{V} . We let ∂_i denote the directional derivative with respect to e_i .

(iii) We first extend v_0 to an element u of $C^{r+1}\Lambda^k(\mathbb{V})$ by putting $u(x+y) = v_0(x)$ for $x \in T$ and $y \in U$. Subtracting this extension we are left with the case $v_0 = 0$ and $\text{pu}_T v_1 = 0$. To avoid clutter we denote $v = v_1$.

(iv) Suppose v is of the form: $v = w w_T \wedge w_U$ with $w_U = f_{j_1} \wedge \dots \wedge f_{j_l}$ (with $l \geq 1$ distinct indices in J), $w_T = f_{i_1} \wedge \dots \wedge f_{i_{k+1-l}}$ (with $k+1-l$ distinct indices in I) and w a scalar function on T .

We trivially extend w to \mathbb{V} , which yields an extension of v to a $(k+1)$ -form on \mathbb{V} , which we still denote by v . We put $u = v \lrcorner X$. We write:

$$d(v \lrcorner X) = \sum_i f_i \wedge \partial_i(v \lrcorner X), \quad (10)$$

$$= \sum_{i \in I} f_i \wedge ((\partial_i v) \lrcorner X) + \sum_{j \in J} f_j \wedge (v \lrcorner e_j). \quad (11)$$

The first term here, when restricted to T , is zero. For the second term we have:

$$\sum_{j \in J} f_j \wedge (v \lrcorner e_j) = \sum_{j \in J} f_j \wedge ((-1)^{k+1-l} w w_T \wedge (w_U \lrcorner e_j)) \quad (12)$$

$$= w w_T \wedge \sum_{j \in J} f_j \wedge (w_U \lrcorner e_j), \quad (13)$$

$$= l v. \quad (14)$$

We also remark that $v \lrcorner X$ is zero on T . Dividing $v \lrcorner X$ by l , we have a suitable extension of $(0, v)$.

(v) Now, in general, the condition $\text{pu}_T v_1 = 0$ guarantees that v_1 is a linear combination of forms $w w_T \wedge w_U$ of the above type, all for some $l \geq 1$. \square

This result motivates the following definition.

Definition 1.1. Let $v_0 \in C^0(T) \otimes \text{Alt}^k(\mathbb{V})$ and $v_1 \in C^0(T) \otimes \text{Alt}^{k+1}(\mathbb{V})$. We say that the pair (v_0, v_1) is *admissible* if $d \text{pu}_T v_0 = \text{pu}_T v_1$, where $d \text{pu}_T v_0$ is defined a priori in the sense of distributions.

A differential acting on admissible pairs. Let T be a flat cell in a vector space \mathbb{V} . Suppose that we have subspaces $B^k(T)$ of $C^0(T) \otimes \text{Alt}^k(\mathbb{V})$, such that the exterior derivative on T maps $\text{pu}_T B^k(T)$ into $\text{pu}_T B^{k+1}(T)$. Then we define the following spaces of admissible pairs:

$$A^k(T) = \{(v_0, v_1) \in B^k(T) \oplus B^{k+1}(T) : d \text{pu}_T v_0 = \text{pu}_T v_1\}. \quad (15)$$

We define the following differential:

$$d^k : \begin{cases} A^k(T) & \rightarrow A^{k+1}(T), \\ (v_0, v_1) & \mapsto (v_1, 0). \end{cases} \quad (16)$$

It is well defined, because if (v_0, v_1) is admissible then $d \text{pu}_T(v_1) = d^2 \text{pu}_T v_0 = 0$, so $(v_1, 0)$ is admissible. Moreover we see that $d^{k+1} \circ d^k = 0$.

Lemma 1.3. *The sequence:*

$$A^k(T) \rightarrow A^{k+1}(T) \rightarrow A^{k+2}(T), \quad (17)$$

is exact if and only if the sequence:

$$\text{pu}_T B^k(T) \rightarrow \text{pu}_T B^{k+1}(T) \rightarrow \text{pu}_T B^{k+2}(T), \quad (18)$$

is exact.

Proof. (i) Suppose the second sequence is exact.

Given an admissible $(v_1, 0) \in A^{k+1}(T)$ we have $d \text{pu}_T v_1 = 0$. Choose $v'_0 \in \text{pu}_T B^k(T)$ such that $dv'_0 = v_1$ and then $v_0 \in B^k(T)$ such that $\text{pu}_T v_0 = v'_0$. Then (v_0, v_1) is admissible and maps to $(v_1, 0)$.

(ii) Suppose the first sequence is exact.

Suppose $v'_1 \in \text{pu}_T B^{k+1}(T)$ satisfies $dv'_1 = 0$. Choose $v_1 \in B^{k+1}(T)$ such that $\text{pu}_T v_1 = v'_1$. Then $(v_1, 0) \in B^{k+1}(T)$ and $d(v_1, 0) = 0$. Writing $(v_1, 0) = d(v_0, v_1)$ we get $v_0 \in B^k(T)$ such that (v_0, v_1) is admissible. Then $v'_0 = \text{pu}_T v_0 \in \text{pu}_T B^k(T)$ satisfies $dv'_0 = v'_1$. \square

Poincaré and Koszul operators. We recall some properties of the so-called Poincaré and Koszul operators, used for constructing finite element differential forms in [16] and [1] respectively. For the former, we refer to [18], especially chapter V, but recall the main steps of interest to us.

Suppose that S is a domain. We consider a smooth map $F : [0, 1] \times S \rightarrow S$, and interpret it as a family of maps $F_t = F(t, \cdot) : S \rightarrow S$, for $t \in [0, 1]$, defining a homotopy between F_0 and F_1 . For most $t \in [0, 1]$, we suppose we have a vector field G_t on S such that, for $x \in S$:

$$G_t(F_t(x)) = \partial_t F_t(x). \quad (19)$$

This uniquely defines G_t on S when $F_t : S \rightarrow S$ is a diffeomorphism, and expresses that any curve $F_\bullet(x)$ flows with G_\bullet .

The Poincaré operator associated with F (and G), acting on differential k -forms, is denoted $\mathfrak{p}[F]$ or, when the choice of F is clear, as \mathfrak{p} . It can be written succinctly:

$$\mathfrak{p}[F]u = \int_0^1 F_t^*(u \mathbin{\mathbb{L}} G_t) dt. \quad (20)$$

More explicitly, if u is a k -form:

$$(\mathfrak{p}[F]u)_x(\xi_2, \dots, \xi_k) = \int_0^1 u_{F_t(x)}(\partial_t F_t(x), DF_t(x)\xi_2, \dots, DF_t(x)\xi_k) dt. \quad (21)$$

From Cartan's formula for Lie derivatives, it follows that the Poincaré operator has the property:

$$F_1^* - F_0^* = \mathfrak{p}[F] d + d \mathfrak{p}[F]. \quad (22)$$

Suppose that F_1 is the identity on S and that F_0 is constant. Then the formula gives, for $\mathfrak{p} = \mathfrak{p}[F]$ acting on k -forms with $k \geq 1$:

$$\text{id} = \mathfrak{p} d + d \mathfrak{p}, \quad (23)$$

whereas if u is a function, considered as a 0-form, and the value of F_0 is W , we get:

$$u - u(W) = \mathfrak{p} d u. \quad (24)$$

If now S is a domain in an affine space, which is starshaped with respect to, say W , we may choose F to be defined by:

$$F_t(x) = tx + (1-t)W. \quad (25)$$

Then we may substitute in the above formulas:

$$\partial_t F_t(x) = x - W, \quad G_t(y) = \frac{1}{t}(y - W), \quad \text{and } DF_t(x)(\xi) = t\xi. \quad (26)$$

We denote the associated Poincaré operator as \mathfrak{p}_W . It is defined explicitly on k -forms u by:

$$(\mathfrak{p}_W u)_x(\xi_2, \dots, \xi_k) = \int_0^1 t^{k-1} u_{W+t(x-W)}(x - W, \xi_2, \dots, \xi_k) dt. \quad (27)$$

In an affine space, given a choice of a point W , we may also define directly a vector field X_W by:

$$X_W : x \mapsto x - W. \quad (28)$$

The contraction of a differential form by X_W is called the Koszul operator associated with W and denoted:

$$\kappa_W : u \mapsto \kappa_W u = u \mathbin{\mathbb{L}} X_W. \quad (29)$$

If the choice of W is clear from the context, we may sometimes omit it from the notation.

If u is a k -form which, with respect to some choice of origin W and basis, has components which are homogeneous polynomials of degree r , then from (27) we get:

$$\mathfrak{p}_W u = \frac{1}{k+r} \kappa_W u, \quad (30)$$

which is polynomial and whose components are homogeneous of degree $r+1$.

We are mainly interested in identities (23,24) and knowing that the Poincaré operator maps polynomials to polynomials, increasing degree by only one. Sometimes explicit computations are more handy with the Koszul operator. For composite elements it will be important where we locate W , so as to respect the refinement used.

Remark 1.1. From the above discussion of Poincaré operators, we can derive the identity, on k -forms which are homogeneous polynomials of degree r :

$$(d\kappa_W + \kappa_W d)u = (r+k)u. \quad (31)$$

It is obtained by two somewhat different techniques in section 3.2 of [1].

2 Low order finite element complexes in 2D

On the necessity of composite elements. Suppose we have two coordinates (x, y) . We have data consisting of functions p_0, p_1 on the x -axis which are $C^1(\mathbb{R})$ and $C^0(\mathbb{R})$ respectively and as well as functions q_0, q_1 on the y -axis that are $C^1(\mathbb{R})$ and $C^0(\mathbb{R})$ respectively.

We want to find a function u on \mathbb{R}^2 of class $C^1(\mathbb{R}^2)$ such that $(u, \partial_y u)$ restricts to (p_0, p_1) on the x -axis and $(u, \partial_x u)$ restricts to (q_0, q_1) on the y -axis. There are compatibility conditions at the origin:

$$(p_0(0), \dot{p}_0(0), p_1(0)) = (q_0(0), q_1(0), \dot{q}_0(0)). \quad (32)$$

These are sufficient for the existence of a $C^1(\mathbb{R}^2)$ extension.

However, for extensions of class $C^2(\mathbb{R}^2)$ of the same data, there is an additional constraint, expressing $\partial_x \partial_y u = \partial_y \partial_x u$, namely:

$$\dot{p}_1(0) = \dot{q}_1(0). \quad (33)$$

This remark applies in particular to polynomials. This explains why we prefer to construct spaces in terms of piecewise polynomials: we can then hope to satisfy first order constraints, without adding second order constraints.

We proceed to define four complexes based on the Clough-Tocher element.

A complex of regularity $(2, 1+, 1)$. Let T be a triangle with vertices V_0, V_1 and V_2 . Choose a point W in the interior of T , and subdivide T into three triangles, by drawing edges from W to V_0, V_1 and V_2 . This equips T with a simplicial refinement, which we denote by \mathcal{R} .

The Clough-Tocher element involves a degree of freedom on the edges, which can be taken as the normal derivative at the midpoint. More generally, for each edge E , we consider a linear form on one-forms, which evaluates the one-form in a transverse direction ν_E , at an interior point W_E of the edge. We denote this degree of freedom on 1-forms as μ_E .

Proposition 2.1. *We have an exact sequence:*

$$0 \longrightarrow \mathbb{R} \longrightarrow C^1 P^3 \Lambda^0(\mathcal{R}) \longrightarrow C_d^0 P^2 \Lambda^1(\mathcal{R}) \longrightarrow C^0 P^1 \Lambda^2(\mathcal{R}) \longrightarrow 0 \quad (34)$$

The spaces are, more explicitly, the following:

- $C^1 P^3 \Lambda^0(\mathcal{R})$ consists of piecewise P^3 functions, which are of class $C^1(T)$.
- $C_d^0 P^2 \Lambda^1(\mathcal{R})$ consists of piecewise P^2 one-forms which are $C^0(T)$ with exterior derivative in $C^0(T)$.
- $C^0 P^1 \Lambda^2(\mathcal{R})$ consists of piecewise P^1 two-forms, which are $C^0(T)$.

Moreover these spaces have the following properties:

- $C^1 P^3 \Lambda^0(\mathcal{R})$ has dimension 12. Any element u is determined by the following data:
 - vertices V : one DoF for $u(V)$ and two DoFs for $du(V)$.
 - edges E : one DoF, say $\mu_E(du)$.
- $C_d^0 P^2 \Lambda^1(\mathcal{R})$ has dimension 15. Any element u is determined by:
 - vertices V : two DoFs for $u(V)$ and one DoF for $du(V)$.
 - edges E : two DoFs, tranverse and tangential: $\mu_E(u)$ and $\int_E u$.
- $C^0 P^1 \Lambda^2(\mathcal{R})$ has dimension 4. Any element u is determined by:
 - vertices V : one DoF for $u(V)$.
 - interior T : one DoF, namely the integral $\int_T u$.

Proof. (i) Exactness of the complex can be deduced from the Poincaré operator associated with the inpoint. It maps the spaces one to the other.

(ii) Counting constraints on the space of piecewise polynomials, shows that the dimension of the first space is at least $30 - 18 = 12$. That the dimension is exactly 12 follows from proving the unisolvence, which we take for granted.

It amounts to showing that if both u and its derivatives are 0 on ∂T , then $u = 0$. Such a u can be written $\lambda_W^2 v$ where $v \in C^0 P^1(\mathcal{R})$, where λ_W is the barycentric coordinate map of \mathcal{R} associated with the inpoint W . That $u \in C^1(T)$ should then imply $v = 0$.

(iii) The last space has dimension 4 and the given degrees of freedom are unisolvence.

(iv) Counting constraints on the spaces of piecewise polynomial one-forms, shows that the dimension of the second space is at least $36 - 21 = 15$. If $u \in C_d^0 P^2 \Lambda^1(\mathcal{R})$ has degrees of freedom 0 we write:

$$u = \mathbf{p} \, du + d \, \mathbf{p} \, u. \quad (35)$$

We notice that $du \in C^0 P^1 \Lambda^2(\mathcal{R})$ and has degrees of freedom 0 so $du = 0$. We also notice that $v = \mathbf{p}_W u$ satisfies $dv = u$. Its degrees of freedom are 0 except perhaps the vertex values $v(V)$. They must be the same, because $\int_E dv = 0$ for each edge E . Hence v is constant, so $u = dv = 0$.

This proves unisolvence and that the dimension count is exact. \square

Notice that the degrees of freedom provide commuting interpolators on T . What remains in order to prove that this is a good finite element, is that *inter-element* continuity behaves as expected. On edges the spaces of restrictions from adjacent triangles should be the same.

A minimal complex of regularity $(2, 1+, 1)$. It is also possible, in the previous example, to eliminate the edge degrees of freedom in $C^1P^3\Lambda^0(\mathcal{R})$, by requiring $du \llcorner \nu_E$ to be affine on edge E . This is called a *reduced* HCT element. The transverse edge degree of freedom in $C_d^0P^2\Lambda^1(\mathcal{R})$ is then also eliminated by requiring $u \llcorner \nu_E$ to be affine. Then we see that we have as many degrees of freedom left for the space of 0-forms (namely three times the number of vertices) as for the space of 2-forms (namely the number of vertices plus number of triangles), up to the Euler-Poincaré characteristic of the surface. This can be interpreted as a balancing of the degrees of freedom describing the curl and the divergence of the vector fields.

This complex is minimal, among complexes with this regularity, in a sense which can be made precise in the framework of finite element systems. This is described below, in the last paragraph of §3.

A complex of regularity $(2, 1, 0)$. We may also consider the sequence:

$$0 \longrightarrow \mathbb{R} \longrightarrow C^1P^3\Lambda^0(\mathcal{R}) \longrightarrow C^0P^2\Lambda^1(\mathcal{R}) \longrightarrow P^1\Lambda^2(\mathcal{R}) \longrightarrow 0 \quad (36)$$

The spaces are, more explicitly, the following:

- $C^1P^3\Lambda^0(\mathcal{R})$ consists of piecewise P^3 functions, which are of class $C^1(T)$.
- $C^0P^2\Lambda^1(\mathcal{R})$ consists of piecewise P^2 one-forms which are $C^0(T)$.
- $P^1\Lambda^2(\mathcal{R})$ consists of piecewise P^1 two-forms.

The second space has dimension 20. The last one has dimension 9. Exactness follows from using the Poincaré operator at the inpoint W . A preliminary reasoning shows that $C^0P^2\Lambda^1(\mathcal{R})$ should have 2 degrees of freedom per vertex, 2 per edge and 8 interior ones. However it is not clear what they should be, if one wants the interpolator to commute with the exterior derivative.

A minimal complex of regularity $(2, 1, 0)$. In the last example, the spaces are bigger than necessary. A smaller complex of the form:

$$0 \longrightarrow \mathbb{R} \longrightarrow A^0(T) \longrightarrow A^1(T) \longrightarrow A^2(T) \longrightarrow 0 \quad (37)$$

may be defined as follows. The spaces are:

- $A^0(T)$ is reduced HCT, of dimension 9.
- $A^1(T) = dA^0(T) + \mathfrak{p}_T A^2(T)$, of dimension 9. The degrees of freedom are, at vertices two for the value, and at edges one for the integral.
- $A^2(T) = P^0\Lambda^2(T)$ consists of constant two-forms on T , of dimension 1. The degree of freedom is the integral.

This complex is minimal, among complexes with this regularity, by the remarks made in the last paragraph of §3.

3 Generalized Finite Element Systems

Motivation for finite element systems. To study the examples of the preceding section, some general theorems make the task easier. Moreover, specifying the degrees of freedom a priori can be difficult when one wants to go to higher order polynomials.

If we are given spaces $A^k(T)$ of k -forms on a cell T , we can actually forget about degrees of freedom and just consider the spaces $A^k(T')$ obtained by restriction to the faces T' of T , with the appropriate definition of restriction, adapted to a particular regularity. Two properties turn out to be sufficient, in order to get a nice finite element:

- $\dim A^k(T) = \sum_{T' \triangleleft T} \dim A_0^k(T')$.
- The sequence $A^\bullet(T')$ is exact on each face T' , except at index 0, where the cohomology group has dimensions 1.

When these properties are satisfied we will show that the sequences $A_0^\bullet(T')$ are exact except at index $\dim T'$, where the cohomology group has dimension 1. Then one can define a commuting interpolator by using the so-called harmonic degrees of freedom, described below.

In a cellular complex the spaces $A^k(T')$ defined on faces should be well defined, in the sense that if they are obtained as the spaces of restrictions from a cell T (containing T') to T' , then they should be independent of T .

Definitions related to finite element systems. Let \mathcal{T} be a cellular complex. If T, T' are cells in \mathcal{T} we write $T' \triangleleft T$ to signify that T' is a subcell of T . Given to cells T and T' in \mathcal{T} , their relative orientation is denoted $\mathbf{o}(T, T')$. It is 0 unless T' is a codimension one subcell of T , in which case it is ± 1 . Cellular cochain complex is denoted $\mathcal{C}^\bullet(\mathcal{T})$. Its differential, also called the coboundary map, is denoted $\delta : \mathcal{C}^k(\mathcal{T}) \rightarrow \mathcal{C}^{k+1}(\mathcal{T})$. Its matrix in the canonical basis is given by relative orientations.

All complexes considered in this paper are cochain complexes in the sense that the differential increases the index.

Definition 3.1. A *finite element system* on \mathcal{T} consists of the following data, which includes both spaces and operators:

- We suppose that for each $T \in \mathcal{T}$, and each $k \in \mathbb{Z}$ we are given a vector space $A^k(T)$. For $k < 0$ we suppose $A^k(T) = 0$.
- For every $T \in \mathcal{T}$ and $k \in \mathbb{Z}$, we have an operator $\mathbf{d}_T^k : A^k(T) \rightarrow A^{k+1}(T)$ called *differential*. Often we will denote it just as \mathbf{d} . We require $\mathbf{d}_T^{k+1} \circ \mathbf{d}_T^k = 0$. This makes $A^\bullet(T)$ into a complex.
- Given T, T' in \mathcal{T} with $T' \triangleleft T$ we suppose we have *restriction* maps:

$$\mathbf{r}_{T'T}^k : A^k(T) \rightarrow A^k(T'), \quad (38)$$

subject to:

- $\mathbf{r}_{T'T} \mathbf{d}_T^k = \mathbf{d}_{T'}^k \mathbf{r}_{T'T}$.
- $\mathbf{r}_{T''T} = \mathbf{r}_{T''T'} \mathbf{r}_{T'T}$.

This makes the family $A^\bullet(T)$, for $T \in \mathcal{T}$, into an inverse system of complexes.

- We suppose we have a map $c_T : \mathbb{R} \rightarrow A^0(T)$. It mimicks inclusion of constant scalar functions. We require:
 - For $T \in \mathcal{T}$, $d_T^0 c_T = 0$.
 - If $T' < T$ are cells in \mathcal{T} , $r_{T'T} c_T = c_{T'}$.
- For T a k -dimensional cell in \mathcal{T} we suppose we have an evaluation map $e : A^k(T) \rightarrow \mathbb{R}$. It mimicks integration of k -forms on a k -cell. We suppose that the following formula holds, for $u \in A^{k-1}(T)$:

$$e_T d_T u = \sum_{T' \in \partial T} o(T, T') e_{T'} r_{T'T} u. \quad (39)$$

It's an analogue of Stokes theorem on T .

If \mathcal{T}' is a cellular subcomplex of \mathcal{T} , the spaces $A^k(T)$ with $T \in \mathcal{T}'$ constitute an inverse system. The inverse limits can be identified as:

$$\varprojlim_{T \in \mathcal{T}'} A^k(T) = \{(u_T)_{T \in \mathcal{T}'} \in \bigoplus_{T \in \mathcal{T}'} A^k(T) : T' \triangleleft T \Rightarrow u_{T'} = r_{T'T} u_T\} \quad (40)$$

We notice that, if T is a cell and $\mathcal{S}(T)$ denotes the cellular complex consisting of all the subcells of T in \mathcal{T} , then the restriction maps provide an isomorphism:

$$r : A^\bullet(T) \rightarrow \varprojlim_{T \in \mathcal{S}(T)} A^\bullet(T). \quad (41)$$

For this reason it seems safe to use the notation:

$$A^\bullet(\mathcal{T}') = \varprojlim_{T \in \mathcal{T}'} A^\bullet(T). \quad (42)$$

Another way of formulating (39) is that for any cellular subcomplex \mathcal{T}' , the evaluation provides a cochain morphism:

$$e : A^\bullet(\mathcal{T}') \rightarrow \mathcal{C}^\bullet(\mathcal{T}'). \quad (43)$$

We will later provide conditions under which it induces isomorphisms on cohomology groups, which would be an analogue of de Rham's theorem.

We denote by $A_0^k(T)$ the kernel of the induced map $r : A^k(T) \rightarrow A^k(\partial T)$. We consider that the boundary of a point is empty, so that if T is a point $A_0^k(T) = A^k(T)$.

Definition 3.2. We say that A admits extensions on $T \in \mathcal{T}$, if the restriction map induces a surjection:

$$r : A^\bullet(T) \rightarrow A^k(\partial T). \quad (44)$$

We say that A admit extensions on \mathcal{T} , if it admits extensions on each $T \in \mathcal{T}$.

This notion corresponds to that of *flabby sheaves* (*faisceaux flasques* in French [15]), due to the following.

Proposition 3.1. *The FES A admits extensions on \mathcal{T} if and only if, for any cellular complexes $\mathcal{T}'', \mathcal{T}'$ such that $\mathcal{T}'' \subseteq \mathcal{T}' \subseteq \mathcal{T}$, the restriction $A^\bullet(\mathcal{T}') \rightarrow A^\bullet(\mathcal{T}'')$ is onto.*

In particular if A admits extensions, then, when T' is a subcell of T , the restriction $A^\bullet(T) \rightarrow A^\bullet(T')$ is onto. However this is in general a strictly weaker condition than the extension property. To see this, consider for instance the finite element spaces $A^0(T)$ consisting of P^1 functions on a quadrilateral S , on its edges E and on its vertices V . Then the restriction from $A^0(S)$ to each edge $A^0(E)$ is onto, as are the other restrictions from faces to subfaces, but the restriction from $A^0(S)$ to $A^0(\partial S)$ is not onto, since the latter has dimension 4 but the former had dimension only 3.

Definition 3.3. We say that A^\bullet is exact on T when the following sequences are exact:

$$0 \longrightarrow \mathbb{R} \xrightarrow{c} A^0(T) \xrightarrow{d} A^1(T) \xrightarrow{d} \dots \quad (45)$$

We say that A^\bullet is locally exact on \mathcal{T} when A^\bullet is exact on each $T \in \mathcal{T}$.

Definition 3.4. We say that A is *compatible* when it admits extensions and is locally exact.

de Rham type theorems. The following theorem extends Proposition 5.16 in [7]:

Theorem 3.2. *Suppose that the element system A is compatible. Then the evaluation maps $e : A^\bullet(\mathcal{T}) \rightarrow \mathcal{C}^\bullet(\mathcal{T})$ induces isomorphisms on cohomology groups.*

Proof. The proof of Proposition 5.16 in [7] works verbatim. \square

We also have the following extension of Proposition 5.17 in [7]:

Theorem 3.3. *Suppose that A has extensions. Then A is compatible if and only if the following two conditions hold:*

- For each $T \in \mathcal{T}$ the ("inclusion of constants") map $c : \mathbb{R} \rightarrow A^0(T)$ is injective.
- For each $T \in \mathcal{T}$ the sequence $A_0^\bullet(T)$ has nontrivial cohomology only at index $k = \dim T$, and there the induced map:

$$e : H^k A_0^\bullet(T) \rightarrow \mathbb{R}, \quad (46)$$

is an isomorphism (it is well defined by (39)).

Proof. We suppose $m > 0$ and that the equivalence has been proved for cellular complexes consisting of cells of dimension $n < m$.

Let $T \in \mathcal{T}$ be a cell of dimension m . We suppose that A is compatible on the boundary of T . Since the boundary is $(m - 1)$ dimensional we may apply the above de Rham theorem there.

We write the following diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & A_0^\bullet(T) & \longrightarrow & A^\bullet(T) & \longrightarrow & A^\bullet(\partial T) \longrightarrow 0 \\ & & \downarrow e & & \downarrow e & & \downarrow e \\ 0 & \longrightarrow & \mathcal{C}_0^\bullet(T) & \longrightarrow & \mathcal{C}^\bullet(T) & \longrightarrow & \mathcal{C}^\bullet(\partial T) \longrightarrow 0 \end{array} \quad (47)$$

On the rows, the second map is inclusion and the third arrow restriction. Both rows are short exact sequences of complexes. The vertical map is the de Rham map. The diagram commutes.

We write the two long exact sequences corresponding to the two rows, and connect them by the map induced by the de Rham map.

$$\begin{array}{ccccccccc}
H^{k-1}A^\bullet(T) & \longrightarrow & H^{k-1}A^\bullet(\partial T) & \longrightarrow & H^k A_0^\bullet(T) & \longrightarrow & H^k A^\bullet(T) & \longrightarrow & H^k A^\bullet(\partial T) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
H^{k-1}\mathcal{C}^\bullet(T) & \longrightarrow & H^{k-1}\mathcal{C}^\bullet(\partial T) & \longrightarrow & H^k \mathcal{C}_0^\bullet(T) & \longrightarrow & H^k \mathcal{C}^\bullet(T) & \longrightarrow & H^k \mathcal{C}^\bullet(\partial T)
\end{array} \tag{48}$$

Suppose that (45) is exact. Then the first and fourth vertical maps are isomorphisms. By the induction hypothesis the second and fifth are isomorphisms. By the five lemma, the third one is an isomorphism. This can be stated as announced.

Suppose that the two stated conditions hold. One applies the five lemma to the long exact sequence, and obtains that $A^\bullet(T)$ is exact, except at index 0. The cohomology group of index 0 is one dimensional, and must consist of the constants, by injectivity of their inclusion. \square

Extensions, dimension counts and harmonic interpolation. The following proposition almost exactly reproduces Proposition in [8].

Proposition 3.4. *Suppose that A^k is an element system and that $T \in \mathcal{T}$. Suppose that, for each cell $U \in \partial T$, each element v of $A_0^k(U)$ can be extended to an element u of $A^k(T)$ in such a way that, $r_{UT}u = v$ and for each cell $U' \in \partial T$ with the same dimension as U , but different from U , we have $r_{U'T}u = 0$. Then A^k admits extensions on T .*

Proof. In the situation described in the proposition we denote by $\text{ext}_U v = u$ a chosen extension of v (from U to T).

Pick $v \in A^k(\partial T)$. Define $u_{-1} = 0 \in A^k(T)$.

Pick $l \geq -1$ and suppose that we have a $u_l \in A^k(T)$ such that v and u_l have the same restrictions on all l -dimensional cells in ∂T . Put $w_l = v - r_{\partial T} u_l \in A^k(\partial T)$. For each $(l+1)$ -dimensional cell U in ∂T , remark that $r_{U\partial T} w_l \in A_0^k(U)$, so we may extend it to the element $\text{ext}_U r_{U\partial T} w_l \in A^k(T)$. Then put:

$$u_{l+1} = u_l + \sum_{U : \dim U = l+1} \text{ext}_U r_{U\partial T} w_l. \tag{49}$$

Then v and u_{l+1} have the same restrictions on all $(l+1)$ -dimensional cells in ∂T .

We may repeat until $l+1 = \dim T$ and then u_{l+1} is the required extension of v . \square

Proposition 3.5. *Let A be a FES on a cellular complex \mathcal{T} . Then:*

- We have:

$$\dim A^k(\mathcal{T}) \leq \sum_{T \in \mathcal{T}} \dim A_0^k(T). \tag{50}$$

- Equality holds in (50) if and only if A^k admits extensions on each $T \in \mathcal{T}$.

Proof. The proof in [8] works verbatim. \square

Suppose now that we are discretizing differential forms, say the sequence $H_d^1 \Lambda^\bullet(S)$ or, more precisely, $C_d^0 \Lambda^k(S)$. For each cell T , equip each space of double traces of $C_d^0 \Lambda^k(S)$, with a continuous scalar product $\langle \cdot | \cdot \rangle$, typically a variant of the L^2 product on forms. For a given finite element system A (equipped with double traces for the restrictions), define spaces $\mathcal{F}^k(T)$ of degrees of freedom as follows. For $k = \dim T$:

$$\mathcal{F}^k(T) = \{ \langle \cdot | v \rangle : v \in \ker d|A_0^k(T) \} \oplus \{ \mathbb{R} \int \cdot \}, \quad (51)$$

and for $k \neq \dim T$:

$$\mathcal{F}^k(T) = \{ \langle \cdot | v \rangle : v \in \ker d|A_0^k(T) \} \oplus \{ \langle d \cdot | v \rangle : v \in dA_0^k(T) \}. \quad (52)$$

This is the natural generalization, to the adopted setting, of so-called projection based interpolation, as defined in [11][12]. We call these the *harmonic degrees of freedom*. For compatible finite element systems these degrees of freedom are unisolvent and yield a commuting interpolator $C_d^0 \Lambda^\bullet(S) \rightarrow A^\bullet(\mathcal{T})$, which we call the *harmonic interpolator*. This topic is detailed in §2.4 of [8], see in particular Proposition 2.8 of that paper.

Minimality. Consider a FES A on a cellular complex \mathcal{T} where the top-dimensional cells are domains in a fixed vectorspace \mathbb{V} of dimension n . Suppose furthermore that each certex lies in an n -dimensional cell. Suppose that, when T is a top-dimensional cell, $A^k(T)$ is a space of k -forms, containing the constant ones. If A is compatible then in particular for each vertex $V \in \mathcal{T}$, the restriction from $A^k(T)$ to $A^k(V)$ is onto. Depending on the nature of restriction we deduce:

- If restriction is the pullback, then $A^k(V) = 0$, except for $k = 0$, in which case it has dimension 1.
- If restriction is the trace, then $A^k(V) = \text{Alt}^k(\mathbb{V})$.
- If restriction is the double trace then $A^k(V) = \text{Alt}^k(\mathbb{V}) \oplus \text{Alt}^{k+1}(\mathbb{V})$.

Moreover, if A is compatible, then we must have, for any k -dimensional cell, $\dim \dim A_0^k(T) \geq 1$, by Theorem 3.3.

These considerations provide a lower bound on $\dim A^k(\mathcal{T})$ in view of Proposition 3.5. We will see examples where this lower bound is attained. These examples are then minimal FES.

This paper defines four minimal FES: two in 2D and two in 3D. In each dimension we distinguish between continuous and discontinuous divergence.

The topic of minimal FES is studied in more detail in [6], in the case where the restriction is the pullback. General recipes for constructing small compatible FES within a big compatible FES are provided.

4 High order composite elements in 2D

We use the term *jet* informally. An r -jet corresponds to a Taylor expansion of order r in some vector bundle, which will here be a vector bundle of differential forms. However for the highest order partial derivatives, only a certain combination of them, corresponding to the exterior derivative, will be used. Moreover the jet exists even when a section it should be the expansion of, is not known a priori.

Extension of 1-jets from vertices. We aim to construct elements in dimension 2 but here we start with some more general comments. Let then \mathbb{V} be a vector space of finite dimension and let V be a point in \mathbb{V} .

We are interested in complexes at V of the form:

$$A^k(V) = \text{Alt}^k(\mathbb{V}) \oplus \text{Alt}^{k+1}(\mathbb{V}). \quad (53)$$

Suppose we are given $(v_0, v_1) \in \text{Alt}^k(\mathbb{V}) \oplus \text{Alt}^{k+1}(\mathbb{V})$ at vertex V . Suppose T is a simplex, of arbitrary dimension, containing V . Let λ be the barycentric coordinate on T with respect to vertex V , and let X be the canonical vectorfield $X : x \mapsto x - V$. We extend the data at V to T by the formula:

$$(u_0, u_1) = (\lambda^2 v_0, 2\lambda d\lambda \wedge v_0) + \quad (54)$$

$$\left(\frac{1}{k+1} \lambda^2 w_1 \lrcorner X, \frac{2}{k+1} \lambda d\lambda \wedge (w_1 \lrcorner X) + \lambda^2 w_1 \right). \quad (55)$$

where $w_1 = v_1 - 2d\lambda \wedge v_0$, considered as a constant $(k+1)$ -form on T .

Notice that u_0 is a differential k -form of polynomial degree 3, that u_1 is a $(k+1)$ form of degree 2, that the pair (u_0, u_1) is admissible, and that its restriction to the other vertices of T is 0, in the sence of double-traces. More stongly, the restriction to the face opposite to V in T is 0.

Remark 4.1. If T is a triangle, Proposition 2.1 guarantees that we have extensions from the vertices of T to T , as required in Proposition 3.4, simply by matching degrees of freedom.

Extension of polynomial 1-jets from edges to triangles. Now suppose E is an edge of a triangle T , living in a vector space \mathbb{V} of dimension 2. We wish to extend data on E to T , so as to be able to apply Proposition 3.4 .

Fix p such that $p \geq 3$. We consider the following spaces, for $0 \leq k \leq \dim \mathbb{V}$.

$$A^k(E) = \{(v_0, v_1) \in P^{p-k}(E) \otimes \text{Alt}^k(\mathbb{V}) \oplus P^{p-k-1}(E) \otimes \text{Alt}^{k+1}(\mathbb{V}) : (v_0, v_1) \text{ is admissible}\}. \quad (56)$$

The admissibility condition is non-trivial only for $k = 0$.

We label the vertices of E with 0 and 1, and the third vertex of T is labelled with 2. The barycentric coordinates on T are, accordingly, denoted $\lambda_0, \lambda_1, \lambda_2$.

We suppose we have chosen an inpoint W on T , and we divide T into three triangles by joining W to the vertices of T . The simplicial complex so obtained is denoted \mathcal{R} .

Lemma 4.1. *There is a function $\Phi \in C^1 P^3(\mathcal{R})$ such that:*

$$\mathrm{tr}_{\partial T}(\Phi) = 0, \quad (57)$$

$$\mathrm{tr}_{\partial T}(d\Phi) = \mathrm{tr}_{\partial T}(\lambda_0 \lambda_1 d\lambda_2). \quad (58)$$

Proof. Follows from Proposition 2.1 by matching degrees of freedom. \square

Lemma 4.2. *There is a 1-form $\Psi \in C_d^0 P^2 \Lambda^1(\mathcal{R})$ such that:*

$$\mathrm{tr}_{\partial T}(\Psi) = \mathrm{tr}_{\partial T}(\lambda_0 \lambda_1 d\lambda_1), \quad (59)$$

$$\mathrm{tr}_{\partial T}(d\Psi) = 0. \quad (60)$$

Proof. Follows from Proposition 2.1 by matching degrees of freedom. \square

Suppose we are given $(v_0, v_1) \in A_0^k(E)$ and that we wish to extend it to T . We may extend this data by 0 to all of ∂T .

– *Case $k = 0$.* First we remark that v_0 is of the form:

$$v_0 = w_0(\lambda_1) \lambda_0^2 \lambda_1^2, \quad (61)$$

where $w_0 \in P^{p-4}(E)$. In this form v_0 is trivially extendable to T , as a function $u_0 \in P^p(T)$. Subtracting $(\mathrm{tr}_E u_0, \mathrm{tr}_E du_0)$ from (v_0, v_1) leaves us with data where $v_0 = 0$. Assuming now that $v_0 = 0$, admissibility shows that v_1 is of the form:

$$v_1 = w_1(\lambda_1) \lambda_0 \lambda_1 d\lambda_2, \quad (62)$$

where $w_1 \in P^{p-3}(E)$. Then we extend $(0, v_1)$ to T as the admissible pair:

$$(u_0, u_1) = (u_0, du_0), \quad (63)$$

$$= (w_1(\lambda_1) \Phi, \dot{w}_1(\lambda_1) d\lambda_1 \Phi + w_1(\lambda_1) d\Phi). \quad (64)$$

In our setup it is only u_0 which is of interest on T , but we need the traces of both u_0 and du_0 on ∂T .

Notice also that the constructed extension satisfies $u_0 \in C^1 P^p \Lambda^0(T)$.

– *Case $k = 1$.* We remark that the data is of the form:

$$v_0 = w_0(\lambda_1) \lambda_0 \lambda_1 d\lambda_1 + w_1(\lambda_1) \lambda_0 \lambda_1 d\lambda_2, \quad (65)$$

$$v_1 = w_2(\lambda_1) \lambda_0 \lambda_1 d\lambda_1 \wedge d\lambda_2. \quad (66)$$

With $w_0 \in P^{p-3}(E)$, $w_1 \in P^{p-3}(E)$ and $w_2 \in P^{p-4}(E)$. We essentially extend the three different components separately, but in a precise order.

First, let $\tilde{w}_2 \in P^{p-3}(E)$ denote an antiderivative of w_2 . Put:

$$u_0 = \tilde{w}_2(\lambda_1) d\Phi \in C_d^0 P^{p-1} \Lambda^1(\mathcal{R}). \quad (67)$$

Then:

$$du_0 = w_2(\lambda_1) d\lambda_1 \wedge d\Phi, \quad (68)$$

whose trace is v_1 . This leaves us with the problem of extending data where $w_2 = 0$.

Second, define:

$$u_0 = w_1(\lambda_1)d\Phi + \dot{w}_1(\lambda)d\lambda_1\Phi \in C_d^0 P^{p-1}\Lambda^1(\mathcal{R}). \quad (69)$$

Then $du_0 = 0$, so in particular $\text{tr}_{\partial T} u_0 = 0$. Moreover:

$$\text{tr}_{\partial T} u_0 = w_1(\lambda_1)\lambda_0\lambda_1 d\lambda_2. \quad (70)$$

This leaves us with the problem of extending data where $w_2 = 0$ and $w_1 = 0$.

Third, define:

$$u_0 = w_0(\lambda_1)\Psi \in C_d^0 P^{p-1}\Lambda^1(\mathcal{R}). \quad (71)$$

Then:

$$\text{tr}_{\partial T} u_0 = w_0(\lambda_1)\lambda_0\lambda_1 d\lambda_1. \quad (72)$$

and moreover:

$$\text{tr}_{\partial T}(du_0) = \text{tr}_{\partial T}(\dot{w}_0(\lambda_1)d\lambda_1 \wedge \Psi) + \text{tr}_{\partial T}(w_0(\lambda_1)d\Psi), \quad (73)$$

$$= 0. \quad (74)$$

This completes the extension procedure.

– *Case $k = 2$.* Then $v_1 = 0$ and v_0 is of the form:

$$v_0 = w_0(\lambda_1)\lambda_0\lambda_1 d\lambda_0 \wedge d\lambda_1, \quad (75)$$

for some $w_0 \in P^{p-4}(E)$. We extend v_0 to T as:

$$u_0 = w_0(\lambda_1)d\lambda_0 \wedge \Psi \in C^0 P^{p-2}\Lambda^2(\mathcal{R}). \quad (76)$$

Definition of finite element spaces. On a triangle T , we define a complex of regularity $(2, 1+, 1)$ depending on a parameter $p \geq 3$. The choice $p = 3$ was described previously, in Proposition 2.1, except for the characterization of spaces attached to faces.

We define the following spaces:

- $A^0(T) = C^1 P^p \Lambda^0(\mathcal{R})$.
It consists of the functions which are \mathcal{R} -piecewise in P^p , and which are of class $C^1(T)$.
- $A^1(T) = C_d^0 P^{p-1} \Lambda^1(\mathcal{R})$.
It consists of the 1-forms which are \mathcal{R} -piecewise in P^{p-1} , and which are of class $C^0(T)$ with exterior derivative in $C^0(T)$.
- $A^2(T) = C^0 P^{p-2} \Lambda^2(\mathcal{R})$.
It consists of the 2-forms which are \mathcal{R} -piecewise in P^{p-2} , and which are of class $C^0(T)$.

We analyse this complex as follows. First we notice:

Proposition 4.3. *The following sequence is exact:*

$$0 \longrightarrow \mathbb{R} \longrightarrow A^0(T) \longrightarrow A^1(T) \longrightarrow A^2(T) \longrightarrow 0, \quad (77)$$

The dimensions are:

$$\dim A^0(T) = (3/2)p(p-1) + 3, \quad (78)$$

$$\dim A^1(T) = 3p(p-2), \quad (79)$$

$$\dim A^2(T) = (3/2)(p-1)(p-2) - 2. \quad (80)$$

Proof. (i) Exactness follows from an application of the Poincaré operator associated with the inpoint W .

(ii) For $A^0(T)$, the dimension is given in [13].

(iii) For $A^2(T)$ the space consists of continuous piecewise P^r functions, on a mesh with 4 vertices, 6 edges and 3 triangles, so with $r = p - 2$. Adding the dimensions of the bubblespaces we get:

$$\dim A^2(T) = 4 + 6(r-1) + 3\left(\frac{(r-1)(r-2)}{2}\right) = \frac{3}{2}r(r+1) + 1. \quad (81)$$

(iv) The dimension of $A^1(T)$ can then be deduced from the exactness of (77):

$$\dim A^1(T) = -1 + \dim A^0(T) + A^2(T). \quad (82)$$

This completes the proof. \square

Remark 4.2. The dimensions are those one obtains by the perhaps naïve approach of counting constraints on piecewise-polynomial differential forms.

For instance, for $A^0(T)$ one starts with the space of \mathcal{R} -piecewise polynomials of degree p . It has dimension $(3/2)(p+2)(p+1)$. To be C^1 at W one imposes two equalities of first order jets, which amounts to 6 conditions. Then, on the edges joining W to the vertices, one expresses continuity, knowing we already have continuity at W as well as continuity of the directional derivative at W along the edge. This gives $3(p-1)$ conditions. Finally one expresses continuity of a transverse derivative on the interior edges, knowing that we already have continuity of it at W . This also gives $3(p-1)$ conditions.

This gives a lower bound on the dimension, since we are not certain at this point that the imposed conditions are linearly independent.

Having examined the spaces $A^k(T)$, we now look at what happens on the faces of T :

– *Vertices.* We define, at a vertex V :

- $A^0(V) = \mathbb{R} \oplus \text{Alt}^1(\mathbb{V})$ interpreted as a value and a value of the differential. Its dimension is 3.
- $A^1(V) = \text{Alt}^1(\mathbb{V}) \oplus \text{Alt}^2(\mathbb{V})$ interpreted as a value and a value of its exterior derivative. Its dimension is 3.
- $A^2(V) = \text{Alt}^2(\mathbb{V})$. Its dimension is 1.

Notice, in view of Lemma 1.3, that we have a well defined complex:

$$0 \longrightarrow \mathbb{R} \longrightarrow A^0(V) \longrightarrow A^1(V) \longrightarrow A^2(V) \longrightarrow 0, \quad (83)$$

where the second arrow $v_0 \mapsto (v_0, 0)$, the third is $(v_0, v_1) \mapsto (v_1, 0)$ and the fourth one is $(v_0, v_1) \mapsto v_1$. Remark that the complex is exact.

– *Edges.* At an edge E we define:

- $A^0(E)$ is the subspace of $P^p(E) \oplus P^{p-1}(E) \otimes \text{Alt}^1(\mathbb{V})$ consisting of admissible pairs (v_0, v_1) . Its dimension is $p + 1 + p = 2p + 1$.
- $A^1(E) = P^{p-1}(E) \otimes \text{Alt}^1(\mathbb{V}) \oplus P^{p-2}(E) \otimes \text{Alt}^2(\mathbb{V})$. Its dimension is $2p + p - 1 = 3p - 1$.
- $A^2(E) = P^{p-2}(E) \otimes \text{Alt}^2(\mathbb{V})$. Its dimension is $p - 1$.

Again, in view of Lemma 1.3, we notice that we have a well defined complex:

$$0 \longrightarrow \mathbb{R} \longrightarrow A^0(E) \longrightarrow A^1(E) \longrightarrow A^2(E) \longrightarrow 0, \quad (84)$$

and that it is exact.

We remark that A defines a finite element system on $\mathcal{S}(T)$, with respect to restriction operators defined by taking double-traces. The crucial missing point is the extension property (flabbiness).

The following result is immediate.

Proposition 4.4. *For any edge E of T , A admits extensions from ∂E to E . Moreover:*

$$\dim A_0^0(E) = 2p + 1 - 2 \cdot 3 = 2p - 5, \quad (85)$$

$$\dim A_0^1(E) = 3p - 1 - 2 \cdot 3 = 3p - 7, \quad (86)$$

$$\dim A_0^2(E) = p - 1 - 2 \cdot 1 = p - 3. \quad (87)$$

And there is nontrivial chomology only at index $k = 1$, where it has dimension 1.

Proposition 4.5. *The Finite Element System A admits extensions from ∂T to T .*

Proof. Using Proposition 3.4 and the results from the preceding paragraph, on extension of jets from vertices and edges. \square

We may summarize our findings as follows:

Theorem 4.6. *The finite element system A defined in this section is compatible.*

5 Tools for composite finite elements

We develop some tools that will be used to define finite element sequences in dimension $n \geq 3$.

Various refinements of simplices. A simplex is a finite non-empty set. Its subsimplices are the non-empty subsets. The geometric realization of a simplex T in a vector space containing the vertices, is its convex hull, denoted $|T|$. Geometric realizations are examples of cells. If T is a simplex with vertices V_0, \dots, V_k we also write $T = [V_0, \dots, V_k]$.

If T is a cell in a cellular complex \mathcal{T} , we denote by $\mathcal{S}_{\mathcal{T}}(T)$ the set of subcells of T in \mathcal{T} , which is also a cellular complex. We denote by $\mathcal{S}_{\mathcal{T}}^k(T)$ the set of those subcells of T which have dimension k . When no confusion is possible we omit the subscript \mathcal{T} . In particular, if T is a simplex the associated simplicial complex is denoted $\mathcal{S}(T)$.

For each simplex T we choose an interior point W_T , called the inpoint of T .

Definition 5.1. Given a simplex T we denote by $\mathcal{R}_m(T)$ the simplicial complex consisting of simplices of the form:

$$[W_{T_k}, W_{T_{k-1}}, \dots, W_{T_0}, V_0, \dots, V_l], \quad (88)$$

such that:

- $T' = [V_0, \dots, V_l]$ is a subsimplex of T of dimension $l \leq m$,
- T_0, \dots, T_{k-1}, T_k are subsimplices of T of dimension at least $m + 1$,
- The simplices are nested as follows: $T' < T_0 < \dots < T_{k-1} < T_k$.

We call $\mathcal{R}_m(T)$ the m -refinement of T .

In particular $\mathcal{R}_0(T)$ is the barycentric refinement of T , at least when the inpoints are chosen to be the isobarycenters. We see that $\mathcal{R}_m(T)$ only uses inpoints of subsimplices of T of dimension at least $m + 1$; subsimplices of T of dimension at most m are not refined. Another way of saying this is that $\mathcal{S}(T)$ and $\mathcal{R}_m(T)$ have the same m -skeleton. For $m \geq \dim T$ we have $\mathcal{R}_m(T) = \mathcal{S}(T)$.

When choosing the inpoints, one is interested in satisfying special properties for adjacent simplices in some simplicial complex, as reviewed in [17]:

- In dimension 2, $\mathcal{R}_1(T)$ is known as a Clough-Tocher split. One is also interested in splits where the inpoints of edges lie on the lines joining the inpoints of the adjacent triangles. Then $\mathcal{R}_0(T)$ is known as a Powell-Sabin split.
- In dimension 3, one is interested in splits where the inpoints on faces lie on the lines joining the inpoints of the two adjacent tetrahedra. Then $\mathcal{R}_1(T)$ is known as a Worsey-Farin split, after [23]. If, in addition, the inpoint on edges lie on a plane containing all the inpoints of the adjacent tetrahedra (i.e. those containing the edge), then $\mathcal{R}_0(T)$ is called a Worsey-Piper split, after [24].
- Actually [23] defines $\mathcal{R}_1(T)$ in arbitrary dimension n and refer to it as generalized Clough-Tocher split. On n -dimensional simplices one chooses arbitrary inpoints. On $(n - 1)$ -dimensional simplices the inpoint is the intersection point with the line joining the inpoints of the two adjacent topdimensional simplices.

- Worsley-Piper splits may be difficult to construct. One example would be to choose, as inpoints, the circumcenters of all subsimplices. A sufficient condition for this choice to yield points in the interior of the simplices, is that simplices are strictly acute. This is quite restrictive.

We note the following:

Lemma 5.1. *We have:*

- For any m , $\mathcal{R}_m(T)$ is a refinement of $\mathcal{R}_{m+1}(T)$.
- If U is a subsimplex of T then:

$$\mathcal{R}_m(T') = \{T' \in \mathcal{R}_m(T) : |T'| \subseteq |U|\}. \quad (89)$$

Alignments in meshes. Already on a triangular mesh in dimension 2, continuity requirements involving derivatives, enforced on piecewise polynomials, may produce complicated spaces. The dimension will in general depend for instance on alignments of edges arriving at vertices. The following result pertains to one such situation.

Suppose S is a two-dimensional vectorspace with a basis (e_1, e_2) . The basis vectors divide S into four sectors, as follows. For the four possibilities of choices of signs $a, b \in \{+, -\}$ we consider the sectors:

$$T_{ab} = \{x_1 e_1 + x_2 e_2 : ax_1 \geq 0 \text{ and } bx_2 \geq 0\}. \quad (90)$$

We consider differential forms, which are piecewise polynomials with respect to this subdivision, with various continuity requirements across interfaces.

Proposition 5.2. *We have an exact sequence on S :*

$$0 \longrightarrow \mathbb{R} \longrightarrow C^1 P^2 \Lambda^0 \longrightarrow C^0 P^1 \Lambda^1 \longrightarrow P^0 \Lambda^2 \longrightarrow \mathbb{R} \longrightarrow 0 \quad (91)$$

where, more precisely:

- $C^1 P^2 \Lambda^0$, the space of continuously differentiable piecewise polynomials of degree 2, has dimension 8. The arrow arriving from \mathbb{R} is inclusion of constants. Any element u will be uniquely determined by the values of the following data:
 - the 1-jet at 0, consisting of the function value $u(0)$ and the differential $Du(0)$.
 - the directional second order derivatives at 0, in the four directions $\pm e_1$ and $\pm e_2$, which, by the way, are well defined.
 - the value of the second order derivative $\partial_1 \partial_2 u$, which, it turns out, must be the same in the four sectors.
- $C^0 P^1 \Lambda^1$ has dimension 10. The arrows arriving to and from this space are exterior derivatives.
- $P^0 \Lambda^2$ has dimension 4. The arrow to \mathbb{R} is the following map:

$$u \mapsto u(++) - u(-+) + u(--) - u(+-). \quad (92)$$

Here $u(ab)$ stands for the value of the two-form u on T_{ab} , or more precisely $u[ae_1 + be_2](e_1, e_2)$.

Remark 5.1. It seems that, if we have just four sectors, without alignments of the edges then the sequence:

$$0 \longrightarrow \mathbb{R} \longrightarrow C^1 P^2 \Lambda^0 \longrightarrow C^0 P^1 \Lambda^1 \longrightarrow P^0 \Lambda^2 \longrightarrow 0 \quad (93)$$

is exact and $C^1 P^2 \Lambda^0$ has dimension only 7.

The situation is reminiscent of [20], which is interested in the last part of the complex, for polynomials of higher order.

Some spaces of piecewise polynomials on simplexes. We first recall:

Proposition 5.3. *Suppose $T = [V_0, \dots, V_n]$ is an oriented simplex of dimension n .*

Suppose that u is a constant n -form on T . Then:

$$\int_T u = \frac{1}{n!} u(V_1 - V_0, V_2 - V_0, \dots, V_n - V_0). \quad (94)$$

Suppose that u is affine on T and 0 at the vertices V_1, \dots, V_n . Then:

$$\int_T u = \frac{1}{(n+1)!} u[V_0](V_1 - V_0, V_2 - V_0, \dots, V_n - V_0). \quad (95)$$

Let S be a simplex of dimension n . All faces T of S are supposed equipped with an orientation and a chosen inpoint W_T .

We shall prove some results of which the following constitute a first case:

Proposition 5.4. *We have the following:*

- *Suppose $u \in C^1 P^2 \Lambda^0(\mathcal{R}_0(S))$, that du is 0 at the vertices of S , and that u has the same value at all vertices of S . Then u is constant on S .*
- *Suppose $u \in C^0 P^1 \Lambda^1(\mathcal{R}_0(S))$ and that $du = 0$. If u is 0 at the vertices of S and the pullback of u to 1-dimensional faces of S has integral 0, then $u = 0$.*

Proof. By induction on $\dim S$. For $\dim S = 0$ there is nothing to prove. Supposing now $n \geq 1$ and that the result has been proved for simplexes S with $\dim S < n$ we proceed as follows, supposing $\dim S = n$.

(i) Choose $u \in C^1 P^2 \Lambda^0(\mathcal{R}_0(S))$ and suppose that $du = 0$ at the vertices. On any $(n-1)$ -face of S the pullback of u is constant by the induction step. Hence u is constant on ∂T . Substractiong this constant, we may suppose that $\text{tr}_{\partial T} u = 0$. Let λ_S be the barycentric coordinate on S attached to the inpoint. We can write $u = \lambda_S v$ for some $v \in C^0 P^1 \Lambda^0(\mathcal{R}_0(S))$. The condition that $u \in C^1(S)$ then lets one conclude.

(ii) Choose $u \in C^0 P^1 \Lambda^1(\mathcal{R}_0(S))$ such that $du = 0$ on S , u is 0 at the vertices of S and the pullback of u to 1-dimensional faces of S has integral 0. Write $u = dv$ with $v \in C^1 P^2 \Lambda^0(\mathcal{R}_0(S))$. We have that dv is zero at vertices. Moreover v has the same values at all vertices, by the one-dimensional Stokes. By the preceding result v is constant, so $u = 0$. \square

Proposition 5.5. *Let S be a simplex with $\dim S \geq 1$. Choose $k \geq 1$. Suppose $u \in C^0 P^1 \Lambda^k(\mathcal{R}_{k-1}(S))$ and that $du = 0$. If u is 0 at the vertices of S and the pullback of u to k -dimensional faces of S has integral 0, then $u(W_S) = 0$.*

Proof. (i) If $k = 1$ the result was proved in the preceding two propositions.

(ii) We suppose now that $k \geq 2$.

For any $(k-1)$ -face U of S define (the real number):

$$c_U = \int_{[W_S, U]} u. \quad (96)$$

This defines a cochain $c_\bullet \in \mathcal{C}^{k-1}(S)$.

– Suppose first $k < n$. We let T be a k -face of S and write, using $du = 0$ and Stokes:

$$0 = \int_{[W_S, T]} du, \quad (97)$$

$$= \sum_{U \in \mathcal{S}^{k-1}(T)} o([W_S, T], [W_S, U]) \int_{[W_S, U]} u, \quad (98)$$

$$= \sum_{U \in \mathcal{S}^{k-1}(T)} o(T, U) \int_{[W_S, U]} u, \quad (99)$$

because the k -faces of $[W_S, T]$ are those containing W_S , in addition to T , where the integral of u is 0 by hypothesis.

This identity can be rewritten, in terms of the simplicial coboundary operator:

$$\delta c_\bullet = 0 \in \mathcal{C}^k(S). \quad (100)$$

– For $k = n$ this identity also holds, and just expresses that $\int_S u = 0$.

(iii) For each vertex V of S , let α_V denote the barycentric coordinate of W_S in S .

Let T be a $(k-2)$ -face of S and denote its vertices V_0, \dots, V_{k-2} . We write, using that u is 0 at vertices of S , and summing over vertices V in S not in T :

$$\sum_{V \notin T} \alpha_V \int_{[W_S, T, V]} u \quad (101)$$

$$= \frac{1}{n!} \sum_{V \notin T} \alpha_V u[W_S](V_0 - W_S, \dots, V_{k-2} - W_S, V - W_S), \quad (102)$$

$$= \frac{1}{n!} u[W_S](V_0 - W_S, \dots, V_{k-2} - W_S, \sum_{V \notin T} \alpha_V (V - W_S)). \quad (103)$$

Then we may substitute:

$$\sum_{V \notin T} \alpha_V (V - W_S) = - \sum_{V \in T} \alpha_V (V - W_S), \quad (104)$$

which gives:

$$\sum_{V \notin T} \alpha_V \int_{[W_S, T, V]} u = 0. \quad (105)$$

This identity can be written:

$$\sum_{U \in \mathcal{S}^{k-1}(S)} \alpha_{U \setminus T} o(U, T) c_U = 0. \quad (106)$$

(iv) If it weren't for the weights $\alpha_{U \setminus T}$, this identity would be $\delta' c_\bullet = 0$, where $\delta' : \mathcal{C}^{k-1}(S) \rightarrow \mathcal{C}^{k-2}(S)$ is the boundary operator, whose matrix in the canonical basis is the transpose of the matrix of δ . Since $\delta c_\bullet = 0$ and $\mathcal{C}^\bullet(S)$ is exact (at index $k-1 \geq 1$), we would conclude immediately that $c_\bullet = 0$.

To account for the weights defined by α , we define, on any subsimplex T of S :

$$\alpha_T = \prod_{V \in \mathcal{S}^0(T)} \alpha_V, \quad (107)$$

and rewrite (106) as:

$$\sum_{U \in \mathcal{S}^{k-1}(S)} (\alpha_U / \alpha_T) \circ(U, T) c_U = 0. \quad (108)$$

Let α_l be the operator $\mathcal{C}^l(S) \rightarrow \mathcal{C}^l(S)$, whose matrix in the canonical basis is diagonal, with entry α_T at index (T, T) , $T \in \mathcal{S}^l(S)$. We obtain:

$$(\alpha_{k-2})^{-1} \delta' \alpha_{k-1} c_\bullet = 0, \quad (109)$$

hence:

$$\delta' \alpha_{k-1} c_\bullet = 0. \quad (110)$$

(v) Now, since $\delta c_\bullet = 0$, we can choose $d_\bullet \in \mathcal{C}^{k-2}(S)$ such that $\delta d_\bullet = c_\bullet$. We have $\delta' \alpha_{k-1} \delta d_\bullet = 0$. Since α_{k-1} is positive definite, we conclude $c_\bullet = 0$.

(vi) Since u is 0 at vertices, for any $(k-1)$ -simplex $U = [V_1, \dots, V_k]$ we have:

$$0 = c_U = (k+1)! u[W_S](V_1 - W_S, \dots, V_k - W_S). \quad (111)$$

There are sufficiently many such $(k-1)$ -simplexes to conclude that $u[W_S] = 0$. \square

Proposition 5.6. *Suppose $u \in C^0 P^1 \Lambda^k(\mathcal{R}_k(S))$ and that du is constant on S . If u is 0 at the vertices of S then u is 0 everywhere.*

Proof. (i) For $k = 0$ the claim is just that an affine function is determined by its vertex values. So we suppose $k \geq 1$ from now on.

(ii) We proceed by induction on $\dim S$. Choose $n \geq 1$ and suppose that the proposition has been proved for simplices S with $\dim S < n$. We call this the outer induction hypothesis. Let S be a simplex of dimension $\dim S = n$.

(iii) For any l -face T of S with $l \leq k$, the trace of u on T is in $C^0 P^1(\mathcal{R}_k(T)) \otimes \text{Alt}^k(\mathbb{V})$ and since T is not refined, u is affine on T . Therefore the trace of u is 0.

In particular the pullback of u to any k -face is 0. Therefore, for any $(k+1)$ -face T of S the pullback v of u satisfies $\int_T dv = 0$. The constant du on S has integral 0 on all $(k+1)$ -faces of S . Therefore $du = 0$ on S .

(iv) Suppose we have proved that the pullback of u to l -faces of S is 0, for some l with $n > l \geq k$. We call this the inner induction hypothesis. Let T be an $(l+1)$ -face of S , and let v be the pullback of u to T . From Proposition 5.5 we conclude that $v(W_T) = 0$.

If $l = k$ then we conclude that $v = 0$.

For $l > k$ we need to check that $v(W_{T'}) = 0$ for faces T' of T of dimension m with $k < m \leq l$.

– Case $m = l$: Let T' be an l -face of T . Let w be the pullback to T' of the $(k-1)$ -form $v \mathbb{L}(W_T - W_{T'})$. We have $w \in C^0 P^1 \Lambda^{k-1}(\mathcal{R}_k(T'))$. We also notice that dw is piecewise constant on T' . Cartan's formula shows that dw is the derivative of v in direction $(W_T - W_{T'})$, which is continuous. Therefore dw is constant. By the outer induction hypothesis, $w = 0$. Now we are in the situation that both v and $v \mathbb{L}(W_T - W_{T'})$ have pullback 0 to T' . Therefore $v(W_{T'}) = 0$.

– Case $k < m < l$: If T'' is an m -face of T , for some $k < m < l$, T'' is included in at least two distinct l -faces of T . Since the pullback to these of v is 0, we deduce $v(W_{T''}) = 0$.

We deduce that $v = 0$ on T . This completes the inner induction (on l), which may be followed up to the case $l = n - 1$. There the conclusion is $u = 0$, and this completes the outer induction step (on n). \square

Remark 5.2. In other words the proposition says that if $u \in C^0 P^1 \Lambda^k(\mathcal{R}_k(S))$ and du is constant, then u is affine on S . The reciprocal is trivial.

Proposition 5.7. *Suppose $u \in C^0 P^1 \Lambda^k(\mathcal{R}_{k-1}(S))$, and that $du = 0$. If u is zero at the vertices of S and for any k -face T of S , $\int_T u = 0$, then $u = 0$.*

Proof. We proceed by induction. We suppose that the proposition has been proved for any S of dimension $n - 1$, and let S be a simplex of dimension n .

From Proposition 5.5, we deduce that $u(W_S) = 0$.

For $k = n$ this is enough to conclude that $u = 0$.

Suppose $k < n$. We want to check that $u(W_T) = 0$ for any m -face T of S with $k \leq m < n$. We distinguish two cases for $m = \dim T$:

– Case $m = n - 1$: We know that the pullback of u to T is 0 by the induction hypothesis. Let w be the pullback of $u \mathbb{L}(W_S - W_T)$ to T . Then w is in $C^0 P^1 \Lambda^{k-1}(\mathcal{R}_{k-1}(T))$ and dw is constant. From Proposition 5.6 it follows that w is zero. We conclude that $u(W_T) = 0$.

– Case $k \leq m < n - 1$: Then T is included in two distinct $(n - 1)$ -faces of S , on which the pullback of u is zero. We deduce that $u(W_T) = 0$. \square

6 Finite element spaces in high dimension

A continuous finite element complex. We consider the following spaces, on a simplex S :

$$K^k(S) = \{u \in C^0 P^1 \Lambda^k(\mathcal{R}_{k-1}(S)) : du = 0\}. \quad (112)$$

We let \mathfrak{p}_S denote the Poincaré operator associated with the inpoint of S . We define the space of k -forms:

$$A^k(S) = K^k(S) + \mathfrak{p}_S K^{k+1}(S). \quad (113)$$

We want to prove that this choice provides a good finite element complex, in the sense that it defines a compatible finite element system.

We first notice:

Proposition 6.1. *We have that:*

- *The sum (113) is direct.*
- *The following sequence is exact:*

$$0 \longrightarrow \mathbb{R} \longrightarrow A^0(S) \longrightarrow A^1(S) \longrightarrow \dots \longrightarrow A^n(S) \longrightarrow 0. \quad (114)$$

Proof. Using essentially that the elements of $K^k(S)$ and $K^{k+1}(S)$ have 0 exterior derivative. \square

Proposition 6.2. *On $A^k(S)$ the degrees of freedom consisting of:*

- *values at vertices,*
 - *values of the exterior derivative at vertices,*
 - *integrals on k -dimensional faces of S (for $k \geq 1$),*
- overdetermine an element.*

Proof. Suppose u is an element and that all these degrees of freedom are 0. Applying Proposition 5.7 first to du and then to u , one first gets that $du = 0$ and then that $u = 0$. \square

For $n = \dim S$ and $k \geq 1$, this gives the lower bounds:

$$\dim A^k(S) \leq (n+1) \binom{n}{k} + \binom{n}{k+1} + \binom{n+1}{k+1} = (n+2) \binom{n+1}{k+1}, \quad (115)$$

and:

$$\dim A^0(S) \leq (n+1)^2. \quad (116)$$

To get unsolvence of the degrees of freedom, we would like to prove the converse bounds on dimension.

We do this for the case of a generalized Powell-Sabin split in dimension $n = 3$. We want to prove:

$$\dim A^3(S) = 5, \quad (117)$$

$$\dim A^2(S) = 20, \quad (118)$$

$$\dim A^1(S) = 30, \quad (119)$$

$$\dim A^0(S) = 16. \quad (120)$$

This amounts to:

$$\dim K^3(S) = 5, \quad (121)$$

$$\dim K^2(S) = 15, \quad (122)$$

$$\dim K^1(S) = 15, \quad (123)$$

$$\dim K^0(S) = 1. \quad (124)$$

Proposition 6.3. *The above dimension counts, in dimension $n = 3$, are correct.*

Proof. (i) For $K^3(S)$ and $K^0(S)$ it is clear.

(ii) For $K^2(S)$ we can get the lower bound as follows: The tetrahedron and its faces are each equipped with an inpoint. So $C^0 P^1 \Lambda^2(\mathcal{R}_1(S))$ has dimension

$(1 + 4 + 4) \cdot 3 = 27$. On the other hand there are 12 subtetrahedra on which we enforce one condition. So $\dim K^2(S) \geq 27 - 12 = 15$.

(iii) For $K^1(S)$ all faces and edges are refined, so $C^0P^1\Lambda^1(\mathcal{R}_0(S))$ has dimension 45. To enforce on an element u of $C^0P^1\Lambda^1(\mathcal{R}_0(S))$, that $du = 0$, we use that du is constant on each of the 24 small tetrahedron of $\mathcal{R}_0(S)$. Therefore it is enough to enforce the pullback to be zero on a set of triangular faces in $\mathcal{R}_0(S)$, such that each little tetrahedron has three of them in its boundary. We choose these triangles as follows:

- We impose that the pullback of du to the triangles joining the inpoint of the tetrahedron, the inpoint of a face and a vertex should be zero. These are 3 conditions per face, and there are 4 faces.

- For each edge, the inpoints of the tetrahedron, the two adjacent triangular faces, and the edge itself are coplanar, by the choice of split. So we may use Proposition 5.2 to impose only 3 conditions, rather than 4. There are 6 edges.

This gives $\dim K^1(S) \geq 45 - 4 \cdot 3 - 6 \cdot 3 = 15$. \square

In arbitrary dimension n we can still be precise about the last two spaces in the complex, which are those relevant for Stokes.

Proposition 6.4. *The given degrees of freedom on $A^{n-1}(T)$ and $A^n(T)$ are unisolvent. The dimensions are $\dim A^{n-1}(S) = (n+1)(n+2)$ and $\dim A^n(S) = n+2$.*

The associated interpolator commutes with the divergence operator.

This gives a minimal good element for continuous vectorfields with continuous divergence.

Proof. (i) We have $\dim A^n(S) = n+2$, since there are $n+1$ vertices in S and we have added the inpoint of S .

(ii) For $K^{n-1}(S)$ we may estimate its dimension as follows. In the refinement $\mathcal{R}_{n-2}(S)$ there are the $n+1$ vertices of S , the $n+1$ inpoints attached to $(n-1)$ -faces, and one inpoint in S . This gives:

$$\dim C^0P^1\Lambda^{n-1}(\mathcal{R}_{n-2}(S)) = n(2(n+1)+1), \quad (125)$$

There are also $(n+1)n$ small n -simplexes, on which we express $du = 0$ as one scalar constraint. This gives:

$$\dim K^{n-1}(T) \geq n(2(n+1)+1) - (n+1)n = n^2 + 2n. \quad (126)$$

(iii) We conclude:

$$\dim A^{n-1}(T) \geq n^2 + 2n + n + 2 = (n+1)(n+2). \quad (127)$$

(iv) Since these lower bounds coincide with the number of degrees of freedom, and these are overdetermining, the degrees of freedom are unisolvent, and the dimension count follows. \square

For general n , the analysis of the complex at lower indices seems more complicated, say for the space $A^1(T)$.

Behavior on faces. We are now interested in determining the restrictions to the faces of S , of the spaces $A^k(S)$. This is important for the inter-element continuity of fields. It is also inherent to the framework of finite element system, which encodes the inter-element continuity by taking an inverse limit.

For this purpose, some alternative characterisations of $A^k(S)$ are sometimes useful. We let κ_S denote the Koszul operator associated with the inpoint of S . We have:

$$\mathfrak{p}_S K^{k+1}(S) = \kappa_S K^{k+1}(S). \quad (128)$$

It follows that:

$$A^k(S) = K^k(S) + \kappa_S K^{k+1}(S). \quad (129)$$

We also have the alternative characterization:

Proposition 6.5. *We have:*

$$A^k(S) = \{u \in C^0 P^2 \Lambda^k(\mathcal{R}_{k-1}(S)) : du \in C^0 P^1 \Lambda^{k+1}(\mathcal{R}_k(S)) \text{ and} \quad (130)$$

$$u - \kappa_S du \in C^0 P^1 \Lambda^k(\mathcal{R}_{k-1}(S))\}, \quad (131)$$

Proof. Using (23). \square

Proposition 6.6. *For any element u of $K^k(S)$, if T is face of S , then for any face U of S with $T \subseteq U \subseteq S$, the pullback of $u \mathbb{L}(W_S - W_U)$ to T is affine.*

Proof. It suffices to show that the pullback of $u \mathbb{L}(W_S - W_U)$ to U is affine. Let v be the pullback of $u \mathbb{L}(W_S - W_U)$ to the simplex $[W_S, U]$. We have that dv is piecewise constant and continuous, as the Lie derivative of u along $W_S - W_U$. Hence dv is constant. We have $\text{pu}_U v \in C^0 P^1 \Lambda^k(\mathcal{R}_k(U))$ and may apply Remark 5.2. \square

Proposition 6.7. *For any element u of $A^k(S)$, if T is face of S , then for any face U of S with $T \subseteq U \subseteq S$, the pullback of $(u - \kappa_T du) \mathbb{L}(W_S - W_U)$ to T is affine.*

Proof. (i) We put $v = du$. We first examine the pullback of $(u - \kappa_U v) \mathbb{L}(W_S - W_U)$ to U . We have:

$$\begin{aligned} (u - \kappa_U v) \mathbb{L}(W_S - W_U) &= (u - \kappa_S v + \kappa_S v - \kappa_U v) \mathbb{L}(W_S - W_U), \\ &= (u - \kappa_S v) \mathbb{L}(W_S - W_U) - \\ &\quad (v \mathbb{L}(W_S - W_U)) \mathbb{L}(W_S - W_U). \end{aligned} \quad (132)$$

The term on the last line is 0. Since $u - \kappa_S v \in K^k(S)$ we may apply the preceding proposition to it. We deduce that the pullback of $(u - \kappa_U v) \mathbb{L}(W_S - W_U)$ to U is affine.

(ii) Now on T we write:

$$u - \kappa_T v = u - \kappa_U v + v \mathbb{L}(W_U - W_T). \quad (133)$$

In the right hand side, we remark that $(u - \kappa_U v) \mathbb{L}(W_S - W_U)$ has a pullback to T which is affine, by the preceding point. Then we consider $w = v \mathbb{L}(W_U - W_T) \mathbb{L}(W_S - W_U)$. From the preceding proposition $v \mathbb{L}(W_S - W_U)$ is affine when pulled back on U . Hence w pulled back to T is also affine. \square

On lower dimensional cells we can define first:

$$M^k(T) = K^k(T) + \kappa_S K^{k+1}(T). \quad (134)$$

We have:

Proposition 6.8. *For any simplexes $T \subseteq U$ in $\mathcal{S}(S)$, the pullback operator gives a map $\text{pu}_T : M^k(U) \rightarrow M^k(T)$.*

Proof. We use the characterization (130) which applies also to $M^k(T)$.

Choose $u \in M^k(U)$ and put $v = \text{pu}_T u$. We have:

$$v - \kappa_T dv = v - \text{pu}_T(du \mathbin{\mathbb{L}} X_T), \quad (135)$$

$$= \text{pu}_T(u - \kappa_U du) - \text{pu}_T(du \mathbin{\mathbb{L}} (W_U - W_T)). \quad (136)$$

The first term in this difference is in $M^k(T)$ by the characterization (130) applied to $M^k(U)$ and $M^k(T)$. The second term is affine on T , by applying Proposition 6.6 to $du \in K^{k+1}(U)$, so it's also in $M^k(T)$. \square

Hence M defines a finite element system with respect to pull-backs. However this is not the restriction operator that interests us for the Stokes equation.

It seems useful to define:

$$\mathbb{W}_T = \text{span}\{W_S - W_U : U \in \mathcal{S}(S) \text{ and } T \subseteq U \subseteq S\}. \quad (137)$$

Motivated by the above considerations we define, for any simplex $T \in \mathcal{S}(S)$:

$$\begin{aligned} A^k(T) = \{ & (u, v) \in C^0 P^2(\mathcal{R}_{k-1}(T)) \otimes \text{Alt}^k(\mathbb{V}) \oplus C^0 P^1(\mathcal{R}_k(T)) \otimes \text{Alt}^{k+1}(\mathbb{V}) : \\ & (u, v) \text{ is admissible and } \text{pu}_T u \in M^k(T) \text{ and} \\ & \forall Y \in \mathbb{W}_T \quad \text{pu}_T(v \mathbin{\mathbb{L}} Y) \text{ and } \text{pu}_T((u - \kappa_T v) \mathbin{\mathbb{L}} Y) \text{ are affine.} \}. \end{aligned} \quad (138)$$

When T is a vertex V this definition reduces to:

$$A^k(V) = \text{Alt}^k(\mathbb{V}) \oplus \text{Alt}^{k+1}(\mathbb{V}). \quad (139)$$

Proposition 6.9. *The spaces $A^k(T)$ constitute a finite element system, with respect to restrictions which are double-traces and differential (16).*

Proof. That restrictions map from $A^k(S)$ to $A^k(T)$ was proved in the preceding three propositions.

That they also map from $A^k(U)$ to $A^k(T)$ when $T \subseteq U \subseteq S$ follows from similar arguments.

Stability under the differential is straightforward. \square

Proposition 6.10. *Suppose $T \in \mathcal{S}(S)$ is not a vertex. If $k = \dim T$ we have:*

$$\dim A_0^k(T) \leq 1. \quad (140)$$

If $k \neq \dim T$, $A_0^k(T) = 0$.

Proof. Suppose $(u, v) \in A_0^k(T)$ and that, in case $k = \dim T$, we have $\int_T \text{pu}_T u = 0$.

By Proposition 6.2 we get $\text{pu}_T u = 0$ and $\text{pu}_T v = 0$.

Then we get that, whenever $Y \in \mathbb{W}_T$, $\text{pu}_T(v \mathbin{\mathbb{L}} Y) = 0$ and $\text{pu}_T(u \mathbin{\mathbb{L}} Y) = 0$, since they are affine and have trace 0 on ∂T .

Since $\mathbb{W}_T + \text{vect } T = \mathbb{V}$, the two conditions above give $u = 0$ and $v = 0$. \square

Theorem 6.11. *The finite element system A is compatible, when $n = 3$, and the split is Powell-Sabin/Worsey-Piper.*

Proof. From Propositions 6.3 and 6.10 we get by computing:

$$\dim A^k(S) \geq \sum_{T \in \mathcal{S}(S)} \dim A_0^k(T). \quad (141)$$

Then Proposition 3.5 shows that equality holds and that the finite element system is flabby. In particular $\dim A_0^k(T) = 1$ for $k = \dim T$, and the integral provides an isomorphism to \mathbb{R} .

The cohomology of the sequence $A_0^\bullet(T)$ is then trivially determined. One concludes by Theorem 3.3. \square

One could also check:

$$\{\text{pu}_T u : (u, v) \in A^\bullet(T)\} = M^k(T), \quad (142)$$

and deduce from there that the sequences $A^\bullet(T)$ resolve \mathbb{R} by Lemma 1.3.

Remark 6.1. A crucial question, to address the case of general n , is whether one has $\mathbb{W}_T \cap \text{vect } T = 0$. It seems that the condition that the following sums are direct:

$$\mathbb{W}_T \oplus \text{vect } T = \mathbb{V}, \quad (143)$$

captures the sort of alignment conditions one needs to impose. Even though \mathbb{W}_T was defined in terms of the choices of inpoints in S , when several n -dimensional simplices meet at T , they should determine the same \mathbb{W}_T .

Branching into Whitney forms. Consider the case of arbitrary $\dim S = n$. Let $\Lambda^k(S)$ denote the space of constant k -forms on S . Fix an index $\ell \in [0, n]$. Instead of (113) we define:

$$A^k(S) = \begin{cases} K^k(S) + \mathfrak{p}_S K^{k+1}(S), & k < \ell, \\ K^k(S) + \mathfrak{p}_S \Lambda^{k+1}(S), & k = \ell, \\ \Lambda^k(S) + \mathfrak{p}_S \Lambda^{k+1}(S) & k > \ell. \end{cases} \quad (144)$$

In this definition we recognize $\Lambda^k(S) + \mathfrak{p}_S \Lambda^{k+1}(S)$ as the space of Whitney k -forms on S , henceforth denoted $\mathfrak{W}^k(S)$. Its canonical choice of degrees of freedom consists of integrals on k -dimensional faces.

Proposition 6.12. *We have that:*

- The sum (113) is direct.
- The following sequence is exact:

$$0 \longrightarrow \mathbb{R} \longrightarrow A^0(S) \longrightarrow A^1(S) \longrightarrow \dots \longrightarrow A^n(S) \longrightarrow 0. \quad (145)$$

The only new space in the above sequence is the one attached to the index ℓ : before ℓ we have the space studied in the previous paragraph and after ℓ we have Whitney forms.

Proposition 6.13. *On $A^\ell(S)$ the degrees of freedom consisting of:*

- evaluation at vertices,
 - integrals of pullback to ℓ -dimensional faces of T ,
- overdetermine an element.*

Proof. If $u \in A^\ell(S)$ has all its degrees of freedom equal to 0, then one checks first that $du = 0$, from the theory of Whitney forms. Then one deduces that $u = 0$ from Proposition 5.7. \square

To get a finite element system we define first:

$$N^\ell(T) = \{u \in C^0 P^2 \Lambda^k(\mathcal{R}_{k-1}(T)) : du \in \Lambda^{k+1}(T) \text{ and} \quad (146)$$

$$u - \kappa_T du \in C^0 P^1 \Lambda^k(\mathcal{R}_{k-1}(T))\}. \quad (147)$$

For $T \in \mathcal{S}(S)$ we then put:

$$A^\ell(T) = \{u \in C^0 P^2(\mathcal{R}_{k-1}(T)) \otimes \text{Alt}^k(\mathbb{V}) : \text{pu}_T u \in N^k(T) \text{ and} \\ \forall Y \in \mathbb{W}_T \quad \text{pu}_T(u \lrcorner Y) \text{ is affine.}\}. \quad (148)$$

For $k < \ell$ one uses the previously defined spaces. For $k > \ell$ one uses Whitney forms. This gives a finite element system.

For $\ell = n - 1$, which is perhaps the most interesting case, from the point of view of Stokes equation, we are able to prove unisolvence:

Proposition 6.14. *The given degrees of freedom on $A^{n-1}(T)$ and $A^n(T)$ are unisolvent. The dimensions are $\dim A^{n-1}(S) = (n+1)^2$ and $\dim A^n(S) = 1$.*

The associated interpolator commutes with the divergence operator.

This gives a minimal good element for continuous vectorfields with discontinuous divergence.

Proof. (i) We have $\dim A^n(S) = 1$, since it consists of the constants.

(ii) The proof of Proposition 6.4 gives the lowerbound:

$$\dim A^{n-1}(S) \geq n(n+2) + 1 = (n+1)^2. \quad (149)$$

which is the number of degrees of freedom defined in Proposition 6.13. \square

Outlook

We finish with some points that merit further investigation, and which we hope to address in a not too distant future:

- We have not included error estimates, but, given that we have defined natural degrees of freedom, we believe these could be obtained by combining techniques developed for HCT (e.g. [10] §46) with general techniques developed for FES (especially in [7]).
- A first natural extension of the present work, would be to define spaces with high approximation order in arbitrary dimension, in particular high order elements for Stokes in dimension 3.
- It is also possible to use the framework of (generalized) FES to describe the complex consisting of the Morley element, the Crouzeix-Raviart element and the piecewise constants (see e.g. [2]). A general framework to discuss many existing non-conforming complexes is within reach.
- The examples discussed in this paper consist of differential forms on domains in a vector space. It seems possible also to extend the techniques to manifolds. This would provide a new method, to solve say the shallow water equations on the sphere.

Acknowledgements

SHC is supported by the European Research Council through the FP7-IDEAS-ERC Starting Grant scheme, project 278011 STUCCOFIELDS.

KH is supported by the China Scholarship Council (CSC), project 201506010013 and by the European Research Council through the FP7-IDEAS-ERC Advanced Grant scheme, project 650138 FEEC-A. The stimulating collaborations are achieved during his visit at University of Oslo (UiO) since September 2015. He is grateful for the kind hospitality and support of UiO.

References

- [1] D. N. Arnold, R. S. Falk, and R. Winther. Finite element exterior calculus, homological techniques, and applications. *Acta Numer.*, 15:1–155, 2006.
- [2] S. C. Brenner. Forty years of the Crouzeix-Raviart element. *Numer. Methods Partial Differential Equations*, 31(2):367–396, 2015.
- [3] S. H. Christiansen. Stability of Hodge decompositions in finite element spaces of differential forms in arbitrary dimension. *Numer. Math.*, 107(1):87–106, 2007. [preprint at arXiv:1007.1120].
- [4] S. H. Christiansen. A construction of spaces of compatible differential forms on cellular complexes. *Math. Models Methods Appl. Sci.*, 18(5):739–757, 2008.
- [5] S. H. Christiansen. Foundations of finite element methods for wave equations of Maxwell type. In *Applied Wave Mathematics*, pages 335–393. Springer, Berlin Heidelberg, 2009.
- [6] S. H. Christiansen and A. Gillette. Constructions of some minimal finite element systems. *Math. Model. Numer. Anal.*, 50(3):833–850, 2016. [preprint at arXiv:1504.04670].
- [7] S. H. Christiansen, H. Z. Munthe-Kaas, and B. Owren. Topics in structure-preserving discretization. *Acta Numerica*, 20:1–119, 2011.
- [8] S. H. Christiansen and F. Rapetti. On high order finite element spaces of differential forms. *Math. Comp.*, 85(296):517–548, 2016. [preprint at arXiv:1306.4835].
- [9] S. H. Christiansen and R. Winther. Smoothed projections in finite element exterior calculus. *Math. Comp.*, 77(262):813–829, 2008.
- [10] P. G. Ciarlet. Basic error estimates for elliptic problems. In *Handbook of numerical analysis, Vol. II*, Handb. Numer. Anal., II, pages 17–351. North-Holland, Amsterdam, 1991.
- [11] L. Demkowicz and I. Babuška. p interpolation error estimates for edge finite elements of variable order in two dimensions. *SIAM J. Numer. Anal.*, 41(4):1195–1208 (electronic), 2003.

- [12] L. Demkowicz and A. Buffa. H^1 , $H(\text{curl})$ and $H(\text{div})$ -conforming projection-based interpolation in three dimensions. Quasi-optimal p -interpolation estimates. *Comput. Methods Appl. Mech. Engrg.*, 194(2-5):267–296, 2005.
- [13] J. Douglas, T. Dupont, P. Percell, and R. Scott. A family of C^1 finite elements with optimal approximation properties for various Galerkin methods for 2nd and 4th order problems. *RAIRO Anal. Numér.*, 13(3):227–255, 1979.
- [14] R. S. Falk and M. Neilan. Stokes complexes and the construction of stable finite elements with pointwise mass conservation. *SIAM J. Numer. Anal.*, 51(2):1308–1326, 2013.
- [15] R. Godement. *Topologie algébrique et théorie des faisceaux*. Hermann, Paris, 1973. Troisième édition revue et corrigée, Publications de l’Institut de Mathématique de l’Université de Strasbourg, XIII, Actualités Scientifiques et Industrielles, No. 1252.
- [16] R. Hiptmair. Canonical construction of finite elements. *Math. Comp.*, 68(228):1325–1346, 1999.
- [17] M.-J. Lai and L. L. Schumaker. *Spline functions on triangulations*, volume 110 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 2007.
- [18] S. Lang. *Fundamentals of differential geometry*, volume 191 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1999.
- [19] M. Neilan. Discrete and conforming smooth de Rham complexes in three dimensions. *Math. Comp.*, 84(295):2059–2081, 2015.
- [20] L. R. Scott and M. Vogelius. Norm estimates for a maximal right inverse of the divergence operator in spaces of piecewise polynomials. *RAIRO Modél. Math. Anal. Numér.*, 19(1):111–143, 1985.
- [21] R. Stenberg. Analysis of mixed finite elements methods for the Stokes problem: a unified approach. *Math. Comp.*, 42(165):9–23, 1984.
- [22] N. J. Walkington. A C^1 tetrahedral finite element without edge degrees of freedom. *SIAM J. Numer. Anal.*, 52(1):330–342, 2014.
- [23] A. J. Worsey and G. Farin. An n -dimensional Clough-Tocher interpolant. *Constr. Approx.*, 3(2):99–110, 1987.
- [24] A. J. Worsey and B. Piper. A trivariate Powell-Sabin interpolant. *Comput. Aided Geom. Design*, 5(3):177–186, 1988.